Chapter 10

Counting

We take a tour through several questions in algorithmic and extremal counting of (geometrically defined) combinatorial objects, with emphasis on how they connect to each other in their solutions. Among these problems are (i) counting the number of simplices spanned by $d+1$ points in a finite set of n points in d-space (simplicial depth), (ii) counting the number of facets of the convex hull of n points in d-space, (iii) investigating the minimal number of crossings a drawing of the complete graph with straight line edges in the plane must have, (iv) counting of crossing-free geometric graphs of several types on so-called wheel-sets (point sets in the plane with all but one point extremal),

Notation.

$0 := (0, 0, \dots, 0)$ is the origin in the considered ambient space.

$\mathbb{N}$ is the set of positive integers, $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$.

$\text{conv}(S)$ denotes the convex hull of a given set S in $\mathbb{R}^d$.

$\binom{S}{k}$ denotes the set of all k-element subsets of a given set S.

Sometimes it may be useful to remember

$$\sum_{i=0}^{n-1} \binom{i}{k-1} = \binom{n}{k} = \binom{n-1}{k} + \binom{n-1}{k-1} .$$

Checkpoints are usually simple facts that are put there to check your understanding of definitions or notions, to be answered perhaps in a minute or two (assuming you have indeed absorbed the definition).

10.1 Introduction

Consider a set $P \subseteq \mathbb{R}^d$ and $q \in \mathbb{R}^d$. A set $A \in \binom{P}{d+1}$ is called q-*embracing simplex* if $q \in \text{conv}(A)$.

The simplicial depth, $\text{sd}_q(P)$, of point q relative to P is the number of q-embracing simplices, i.e.

$$\text{sd}_q(P) := \left| \left\{ A \in \binom{P}{d+1} \;\middle|\; p \in \text{conv}(A) \right\} \right| .$$

This notion, among others, is a possible response to the search for a higher-dimensional counterpart of the notion of a median in $\mathbb{R}^1$. Note here that when specialized to $\mathbb{R}^1$, a median is a point of maximum simplicial depth. We will investigate here this notion, asking questions like:

What is the maximum possible simplicial depth a point can have in any set of n *points in general position?*

How efficiently can we compute the simplicial depth of a point?

A second question we want to address is that of the complexity of polytopes in general dimension d.

How many facets can a polytope obtained as the convex hull of n *points have, how few?*

Given n *points, how efficiently can we compute the number of facets? Can we do that asymptotically faster than enumerating these facets (which is a hard enough problem per se)?*

A small caveat, in case you are not familiar with this: We know that a 2-dimensional polytope with n vertices has n facets (here edges), and a 3-dimensional polytope with n vertices has at most $2n-4$ facets (if the vertices are in general position, it is exactly this number). So the number of facets is linear in n. This last fact is not true in higher dimension and we will see what the right bounds are.

We will see that these two types of questions about simplicial depth and number of facets of a polytope are very closely related; in some sense, that we will make very explicit, it is the same question.

On the side, we will also indicate how these connect to other problems as indicated in the abstract.

10.2 Embracing k-Sets in the Plane

In this section we investigate simplicial depth in the plane, and generalize by considering arbitrarily large sets with a given point in their convex hull. This will allow us to introduce some of the technicalities in this simpler (planar) context, and we will see later that the extension to larger sets was unavoidable, even when we are interested in simplicial depth only.

Consider a set $P \subseteq \mathbb{R}^2$, with $0 \notin P$ and $P \mathbin{\dot\cup} \{0\}$ in general position (no three on a line); $n := |P|$. This setting will be implicitly assumed throughout the section. For $k \in \mathbb{N}_0$, we define

$$e_k = e_k(P) := \left| \left\{ A \in \binom{P}{k} \;\middle|\; 0 \in \operatorname{conv}(A) \right\} \right| .$$

We call A with $0 \in \operatorname{conv}(A)$ an *embracing* k*-set* – if $|A| = 3$, we call A an *embracing triangle.*

Checkpoint 10.1. *e_3 is the simplicial depth of 0 in P, i.e. $e_3 = \mathrm{sd}_0(P)$. $e_0 = e_1 = e_2 = 0$, $e_n \in \{0, 1\}$.*

We start a general investigation of $e = (e_0, e_1, \ldots, e_n)$. Bounds and algorithms will follow easily. For a preparatory step consider real vectors $x_{0..n-3} = (x_0, x_1, \ldots, x_{n-3})$, $y_{0..n-2}$ and $z_{0..n-1}$ satisfying

$$e_k = \sum_{i=0}^{n-3} \binom{i}{k-3} x_i\,, \quad \text{for } k \geqslant 3, \tag{10.2}$$

$$e_k = \sum_{i=0}^{n-2} \binom{i}{k-2} y_i\,, \quad \text{for } k \geqslant 2, \text{ and} \tag{10.3}$$

$$e_k = \sum_{i=0}^{n-1} \binom{i}{k-1} z_i\,, \quad \text{for } k \geqslant 1. \tag{10.4}$$

Observe that $x_{0..n-3}$ exists and is uniquely determined by $e_{3..n}$, since

$$\begin{aligned} e_n &= \tbinom{n-3}{n-3} x_{n-3} && \Rightarrow x_{n-3} = e_n \\ e_{n-1} &= \tbinom{n-4}{n-4} x_{n-4} + \tbinom{n-3}{n-4} x_{n-3} && \Rightarrow x_{n-4} = e_{n-1} - (n-3)\underbrace{x_{n-3}}_{e_n} \\ &\vdots \end{aligned}$$

Similarly, this works for $y_{0..n-2}$ and $z_{0..n-1}$. Thus we have

$$e_{3..n} \overset{\text{determine}}{\underset{\text{each other}}{\longleftrightarrow}} x_{0..n-3}, \quad e_{2..n} \overset{\text{determine}}{\underset{\text{each other}}{\longleftrightarrow}} y_{0..n-2}, \quad e_{1..n} \overset{\text{determine}}{\underset{\text{each other}}{\longleftrightarrow}} z_{0..n-1}\,.$$

For now, it is by no means clear what that should help here. Note also that these facts are true for any vector e, we have not used any of the properties of the specific vector we are interested in. They simply describe one of many possible transformations of a given vector.

10.2.1 Adding a Dimension

Another step that comes across unmotivated: Lift the the point set P vertically to a set P' in space, arbitrarily, with the only condition that P' is in general position (no four on a plane).[1] We denote the map by

$$P \ni \quad q = (x, y) \mapsto q' = (x, y, z(q)) \quad \in P' .$$

For an embracing triangle $\Delta = \{p, q, r\}$ in the plane, let β_Δ be the number of points in P' below the plane containing $\Delta' = \{p', q', r'\}$. (Just to avoid confusion: β_Δ clearly depends on the choice of the lifting P'.) Let

$$h_i = h_i(P') := \text{ the number of embracing triangles } \Delta \text{ with } \beta_\Delta = i.$$

Checkpoint 10.5. $\sum_{i=0}^{n-3} h_i = e_3$.

Let us recall here that we are assuming general position for $P \dot\cup \{0\}$.

Lemma 10.6. $0 \in \text{conv}(P) \iff h_0 = h_{n-3} = 1$.

Proof. ($\Leftarrow$) That's obvious, since $h_0 = 1$ means that there is some embracing triangle, and therefore 0 is in the convex hull of P.

($\Rightarrow$) Note that $0 \in \text{conv}(P)$ iff the z-axis (i.e. the vertical line through 0) intersects $\text{conv}(P')$. There are exactly two facets (triangles because of general position) intersected by the z-axis, the bottom one, Δ'_0 has no point in P' below its supporting plane, hence, $\beta_{\Delta_0} = 0$; the top one, Δ'_1 has no point in P' above and hence $n-3$ points in P' below (all but the three points defining the facet), hence, $\beta_{\Delta_1} = n - 3$. Since any triple $\Delta' \subseteq P'$ with all points in P' on one side (above or below) must give rise to a facet, it cannot be hit by the z-axis, unless $\Delta' = \Delta'_0$ or $\Delta' = \Delta'_1$. □

Consider an embracing k-set A and its lifting A'. As observed before, in $\mathbb{R}^3$ the vertical line through 0 will intersect the boundary of $\text{conv}(A')$ in two facets. Consider the top facet – its vertices are liftings of some embracing triangle Δ in the plane. We call this Δ the *witness of* (the embracing property of) A. For how many embracing k-sets is Δ the witness?

For Δ to be witness of an embracing k-set B, we must have $\Delta \subseteq B$ and the remaining $k-3$ points in $B \setminus \Delta$ must be chosen so that $B' \setminus \Delta'$ lies below the plane spanned by Δ'. Hence Δ is witness for exactly $\binom{\beta_\Delta}{k-3}$ embracing k-sets. It follows that

$$e_k = \sum_{\Delta \text{ embracing}} \binom{\beta_\Delta}{k-3} = \sum_{i=0}^{n-3} \binom{i}{k-3} h_i . \tag{10.7}$$

[1]For example, choose the lifting map $(x, y) \mapsto (x, y, x^2 + y^2)$ ... but stay flexible!

That is, the h_i's are exactly the x_i's defined by equations (10.2). As observed before, we thus have

$$e_{3..n} \overset{\text{determine}}{\underset{\text{each other}}{\longleftrightarrow}} h_{0..n-3} := (h_0, h_1, \ldots, h_{n-3})$$

and therefore the vector $h_{0..n-3}$ is independent of the lifting we chose, i.e. $h_i = h_i(P)$.

A few properties emerge. First note that h (consisting of nonnegative integers, each at most $\binom{n}{3} = O(n^3)$, i.e. $O(\log n)$ bits) is a compact way of representing e (with numbers, some may be exponential in n, with $\Omega(n)$ bits). Also, since it is easy to compute the vector h in $O(n^4)$ time[2], we can compute each entry of e_k in $O(n^4)$ time.

Exercise 10.8. *Show*

$$h_0 = 1 \quad \Leftrightarrow \quad 0 \in \operatorname{conv}(P) \quad \Leftrightarrow \quad h_i \geqslant 1 \text{ for } 0 \leqslant i \leqslant n-3 \ .$$

Exercise 10.9. *Assume $0 \in \operatorname{conv}(P)$. (i) What is the minimal possible value of e_3 in terms of $n := |P|$? (Note that this gives a quantified version of Carathéodory's Theorem.) (ii) What is the minimal possible value of e_k, $3 \leqslant k \leqslant n$?*

Exercise 10.10. *What does $\sum_{i=0}^{n-3} 2^i h_i$ count?*

Exercise 10.11. *Show $\sum_{k=3}^{n} (-1)^k e_k = -1$ provided $0 \in \operatorname{conv}(P)$. (Hint: Plug in $\sum_{i=0}^{n-3} \binom{i}{k-3} h_i$ for e_k in this sum and simplify.)*

In a next step we show that the vector h is symmetric.

Lemma 10.12. $h_i = h_{n-3-i}$.

Proof. Define $\hat{h}_i$ in the same way as h_i, except that you count the points *above* (instead of below) the plane through the lifting of an embracing triangle. First, note that $\hat{h}_i = h_{n-3-i}$. And clearly, (with the same witness argument as before)

$$e_k = \sum_{i=0}^{n-3} \binom{i}{k-3} \hat{h}_i \ ,$$

and, therefore,

$$h_i = \hat{h}_i = h_{n-3-i} \ .$$

□

That is, vector $h_{0..n-3}$ is determined by entries $h_0, h_1, \ldots, h_{\lfloor (n-3)/2 \rfloor}$.

Exercise 10.13. *Show $(n-3)e_3 = 2e_4$.*

Exercise 10.14. *Show that if $|P| = 6$, then e_3 determines $e_{3..6}$. How?*

Exercise 10.15. *Show that if $|P|$ is even then e_3 is even.*

[2] With some basics from computational geometry, in $O(n^3)$ time.

10.2.2 The Upper Bound

We have seen in one of the exercises how the relation between e and h can be useful in proving lower bounds on the e_k's. We need two lemmas towards a proof of upper bounds.

The first lemma states that removing a point in P cannot increase h_j.

Lemma 10.16. *For all* $q \in P$, $h_j(P \setminus \{q\}) \leqslant h_j(P)$.

Proof. Which changes happen to h_j as we remove a point q in P?

- *We lose* embracing triangles Δ with j points below (in the lifting), one of which is q. And we lose embracing triangles Δ, where q is a defining point (i.e. $q \in \Delta$).
- *We keep* embracing triangles Δ with j points below, and q above.
- *We gain* embracing triangles Δ with $j+1$ points below, with q one of those below.

Now move q' (in the lifting) vertically above all planes defined by three points in $P' \setminus \{q'\}$. This does not change the values h_i (since, again, h is independent of the lifting), and the case of "We gain" cannot occur. This gives the lemma. □

Lemma 10.17. $\sum_{q\in P} h_j(P \setminus \{q\}) = (n-j-3)h_j(P) + (j+1)h_{j+1}(P)$.

Proof. A contribution to $\sum_{q\in P} h_j(P \setminus \{q\})$ can come only from triangles Δ with $\beta_\Delta = j$ or $\beta_\Delta = j+1$ (relative to the complete set P and a chosen lifting P').

- If $\beta_\Delta = j$, Δ' remains a triangle with j points below, if q is chosen as one of the $(n-3-j)$ points above.
- If $\beta_\Delta = j+1$, Δ' turns into a triangle with j points below, if q is chosen as one of the $(j+1)$ points below.

Hence the lemma. □

Now we recall the previous Lemma 10.16 to bound the sum in Lemma 10.17:

$$\sum_{q\in P} h_j(P \setminus \{q\}) \leqslant n \cdot h_j(P) ,$$

and with this we can derive

$$\begin{aligned} (n-j-3)h_j(P) + (j+1)h_{j+1}(P) &\leqslant n \cdot h_j(P) \\ (j+1)h_{j+1}(P) &\leqslant (j+3)h_j(P) \\ h_{j+1}(P) &\leqslant \frac{j+3}{j+1} h_j(P) . \end{aligned}$$

This bund can be iterated until we reach h_0:

$$h_{j+1}(P) \leqslant \tfrac{j+3}{j+1} h_j(P) \leqslant \tfrac{j+3}{j+1}\,\tfrac{j+2}{j}\, h_{j-1}(P) \leqslant \underbrace{\frac{j+3}{j+1}\,\frac{j+2}{j}\cdots\frac{3}{1}}_{=\binom{j+3}{2}} \underbrace{h_0(P)}_{\leqslant 1} \leqslant \binom{j+3}{2} .$$

Theorem 10.18. *Let* P *be a set of* n *points in general position.*

(i) For all j, $0 \leqslant j \leqslant n-3$,

$$h_j = h_{n-3-j} \quad \text{and} \quad h_j \leqslant \binom{j+2}{2}$$

and hence $h_j \leqslant \min\{\binom{j+2}{2}, \binom{n-1-j}{2}\}$.

(ii)

$$e_3 \leqslant \begin{cases} 2\binom{n/2+1}{3} = \frac{n(n^2-4)}{24} & \text{for } n \text{ even, and} \\ 2\binom{(n+1)/2}{3} + \binom{(n+1)/2}{2} = \frac{n(n^2-1)}{24} & \text{for } n \text{ odd.} \end{cases}$$

Proof. (i) is just a summary of what we have derived so far.

For (ii) we simply plug these bounds into relation (10.7). Suppose first that n is even. Then

$$(h_0, h_1, \ldots, h_{n/2-2}) = (h_{n-3}, h_{n-2}, \ldots, h_{n/2-1})$$

and, therefore,

$$e_3 = \sum_{i=0}^{n-3} h_i = 2\sum_{i=0}^{n/2-2} h_i \leqslant 2\sum_{i=0}^{n/2-2} \binom{i+2}{2} = 2\binom{n/2+1}{3}.$$

Second, if n is odd then

$$(h_0, h_1, \ldots, h_{(n-3)/2}) = (h_{n-3}, h_{n-2}, \ldots, h_{(n-3)/2})$$

with $h_{(n-3)/2}$ appearing in both sequences. Then

$$\begin{aligned} e_3 = \sum_{i=0}^{n-3} h_i &= 2\sum_{i=0}^{(n-3)/2-1} h_i + h_{(n-3)/2} \\ &\leqslant 2\sum_{i=0}^{(n-3)/2-1} \binom{i+2}{2} + \binom{(n+1)/2}{2} \\ &= 2\binom{(n+1)/2}{3} + \binom{(n+1)/2}{2}. \end{aligned}$$

□

There are sets where all these bounds are tight, simultaneously. We find it more convenient to substantiate this claim after some further considerations.

Exercise 10.19. *Show* $e_3 \leqslant \frac{1}{4}\binom{n}{3} + O(n^2)$. *(That is, asymptotically, at most* $1/4$ *of all triangles embrace the origin.)*

Exercise 10.20. *Try to understand the independence of* h *of the actual lifting by observing what happens as you move a single point vertically.*

While we have successfully obtained lower and upper bounds, we will next give a better method for computing the e_k's.

10.2.3 Faster Counting—Another Vector

Call a directed edge $0q$, $q \in P$, an i-*edge*, if i points in P lie to the left of the directed line through $0q$ (directed from 0 to q). Let $\ell_i = \ell_i(P)$ be the number of i-edges of P.

Checkpoint 10.21. $\sum_i \ell_i = n$. *What is the vector* $l = (\ell_0, \ell_1, \ldots, \ell_{n-1})$ *for the case* $0 \notin \text{conv}(P)$*?*

For every nonempty set $A \subseteq P$ with $0 \notin \text{conv}(A)$, $\text{conv}(A)$ has a left and a right tangent from 0. Let $q \in A$ be the touching point of the right tangent. For how many k-element sets $A \subseteq P$ with $0 \notin \text{conv}(A)$ is this point q the right touching point?

Checkpoint 10.22. *This is* $\binom{i}{k-1}$ *if* $0q$ *is an* i*-edge.*

Hence, we have for $1 \leqslant k \leqslant n$:

$$e_k = \underbrace{\binom{n}{k}}_{\sum_{i=0}^{n-1}\binom{i}{k-1}} - \sum_{i=0}^{n-1}\binom{i}{k-1}\ell_i = \sum_{i=0}^{n-1}\binom{i}{k-1}(1-\ell_i)\ . \tag{10.23}$$

We have a combinatorial interpretation of the z_i's in (10.4) and, therefore, numbers ℓ_i satisfying (10.23) are unique.

Exercise 10.24. *Show that* $\ell_i = \ell_{n-1-i}$*. (Hint: Wonder why we chose "left" and not "right".)*

We can compute the vector $l_{0..n-1}$ in $O(n \log n)$ time. For that we rotate a directed line about 0, starting with the horizontal line, say. We always maintain the number of points left of this line, and update this number whenever we sweep over a point $q \in P$. This q may lie ahead of 0 or behind it; depending on this the number increases by 1 or decreases by 1, resp. In this way, with a rotation by 180 degrees, we can compute the "number of points to the left" for every $q \in P$. We need $O(n \log n)$ time to sort the events (encounters with points in P). We initialize the "number to the left" in $O(n)$ time in the beginning, and then update the number in $O(1)$ at each event. This gives $O(n \log n)$ altogether.

Theorem 10.25. *In the plane, the simplicial depth* $\text{sd}_q(P)$ *can be computed in* $O(n \log n)$ *time, provided* $P \,\dot\cup\, \{q\}$ *is in general position.*

Clearly, all entries e_k, $1 \leqslant k \leqslant n$, can be computed based on the vector l. However, keep in mind that the binomial coefficients involved in the sum (10.23) must be determined and that the numbers involved are large (up to n-bits numbers).

Showing that the upper bound in Theorem 10.18 is tight is now actually easy.

If P is the set of vertices of a regular n-gon, n odd, centered at 0, then $\ell_{(n-1)/2} = n$ (and all other ℓ_i's vanish). Therefore,

$$e_3 = \binom{n}{3} - \binom{(n-1)/2}{2} \cdot n = \frac{n(n^2-1)}{24} ,$$

and the case of n odd is shown tight in Theorem 10.18.

For n even, consider the vertices of a regular n-gon centered at 0, and let P be a slightly perturbed set of these vertices so that $P \cup \{0\}$ is in general position. Note that all edges $0q$, $q \in P$, must be $(n/2-1)$- or $(n/2)$-edges. Interestingly, because of the symmetry of the ℓ-vector, we immediately know that $\ell_{n/2-1} = \ell_{n/2} = n/2$ (with all other ℓ_i's vanishing), independent of our perturbation. Now

$$e_3 = \binom{n}{3} - \left(\binom{n/2-1}{2} + \binom{n/2}{2}\right)\frac{n}{2} = \frac{n(n^2-4)}{24} ,$$

and Theorem 10.18 is proven tight also for n even.

A next step is to understand what the possible ℓ-vectors for n points are, and in this way characterize and eventually count all possibilities for l and thus for e.

10.2.4 Characterizing All Possibilities

We start with two observations about properties of l.

Exercise 10.26. *Show that $\ell_{\lfloor (n-1)/2 \rfloor} \geqslant 1$. (There is always a halving edge.)*

Exercise 10.27. *Show that if $\ell_i \geqslant 1$ for some $i \leqslant \lfloor (n-1)/2 \rfloor$, then $\ell_j \geqslant 1$ for all j, $i \leqslant j \leqslant \lfloor (n-1)/2 \rfloor$.*

We summarize our knowledge about l.

Theorem 10.28. *For $n \in \mathbb{N}$, the vector $l_{0..n-1}$ of an n-point set satisfies the following conditions.*

- *All entries are nonnegative inegers.*
- $\sum_{i=0}^{n-1} \ell_i = n$.
- $\ell_i = \ell_{n-1-i}$. *(Symmetry)*
- *If $\ell_i \geqslant 1$ for some $i \leqslant \lfloor (n-1)/2 \rfloor$, then $\ell_j \geqslant 1$ for all j, $i \leqslant j \leqslant \lfloor (n-1)/2 \rfloor$. (remains positive towards the middle)*

Let us call a vector of length n a *legal n-vector* if the conditions of Theorem 10.28 are satisfied. Then (1) is the only legal 1-vector, $(1,1)$ is the only legal 2-vector, and the only legal 3-vectors are $(0,3,0)$ and $(1,1,1)$. The following scheme displays how we derive legal 6-vectors from legal 5-vectors, and how we can derive legal 7-vectors from legal 5- or 6-vectors.

n=5

0	0	5	0	0
0	1	3	1	0
0	2	1	2	0
1	1	1	1	1

add 1 in the middle and split $\longrightarrow$

n=6

0	0	3	3	0	0
0	1	2	2	1	0
0	2	1	1	2	0
1	1	1	1	1	1

add 2 in the middle $\downarrow$

0	0	0	7	0	0	0
0	0	1	5	1	0	0
0	0	2	3	2	0	0
0	1	1	3	1	1	0

n=7, with $>$1 in the middle

insert a 1 in the middle $\downarrow$

0	0	3	1	3	0	0
0	1	2	1	2	1	0
0	2	1	1	1	2	0
1	1	1	1	1	1	1

n=7, with =1 in the middle

Exercise 10.29. *Show that the scheme described, when applied to general n odd, is complete. That is, starting with all legal n-vectors, n odd, we get all legal $(n+1)$-vectors, and from the n- and $(n+1)$-vectors, we get all $(n+2)$-vectors.*

Exercise 10.30. *Show that the number of legal n-vectors is exactly $2^{\lfloor (n-1)/2 \rfloor}$.*

Exercise 10.31. *Show that every legal n-vector is the ℓ-vector of some set of n points in general position.*

With these exercises settled, we have given a complete characterization of all possible ℓ-vectors, thus of all possible e-vectors.

Theorem 10.32. *The number of different e-vectors (or ℓ-vectors) for n points is exactly $2^{\lfloor (n-1)/2 \rfloor}$.*

Exercise 10.33. *Show that $\sum_{i=0}^{j} \ell_i \leqslant j+1$ for all $0 \leqslant j \leqslant \lfloor (n-1)/2 \rfloor$. (Hint: Otherwise, we get into conflict with "remains positive towards the middle").*

10.2.5 Some Add-Ons

We are still missing an interpretation of the y_i's in relations (10.3). We want to leave this as an exercise.

Exercise 10.34. *For a set P of n points in general position, consider the vector*

$$(b_0, b_1, \ldots, b_{n-2})$$

defined by the relations

$$e_k = \binom{n}{k} - \sum_{i=0}^{n-2} \binom{i}{k-2} b_i = \sum_{i=0}^{n-2} \binom{i}{k-2} (n-i-1-b_i) \ ,$$

for $2 \leqslant k \leqslant n$. *Give a combinatorial interpretation of these numbers* b_i, $0 \leqslant i \leqslant n-2$.

Finally, let us investigate how the vectors x, y, and z from relations (10.2), (10.3), and (10.4) connect to each other. Clearly, e_1 and e_2 given, they determine each other. But how? This will allow us to relate the vectors h and l.

Exercise 10.35. *Consider the relations defined in the beginning of this section on* $x_{0..n-3}, y_{0..n-2}$, $z_{0..n-1}$, *and* $e_{1..n}$ *(using* $e_1 = e_2 = 0$*). Then the* y_i*'s are the forward differences of the* x_i, *and the* z_i*'s are the forward differences of the* y_i*'s. Prove this. More concretely, show that*

$$y_i = \begin{cases} -x_0 & i = 0 \\ x_{i-1} - x_i & 1 \leqslant i \leqslant n-3 \\ x_{n-3} & i = n-2 \end{cases}$$

or, equivalently,

$$y_i = x_{i-1} - x_i \quad \text{for all } 0 \leqslant i \leqslant n-2, \text{ with } x_{-1} := x_{n-2} := 0.$$

Show that this entails as well

$$x_i = -\sum_{j=0}^{i} y_j \quad , \text{for } 0 \leqslant i \leqslant n-3.$$

Exercise 10.36. *Prove for vectors* $a_{0..m}$ *and* $b_{0..m}$

$$\forall k, 0 \leqslant k \leqslant m: \quad a_k = \sum_{i=0}^{m} \binom{i}{k} b_i$$

$$\iff \forall i, 0 \leqslant i \leqslant m: \quad b_i = \sum_{k=0}^{m} (-1)^{i+k} \binom{k}{i} a_k \ .$$

Exercise 10.37. *Employing the previous exercise, what does* $h_0 = 1$ *say about* $e_{3..n}$.

The following facts can now be readily derived.

Theorem 10.38.

$$h_i = \binom{i+2}{2} - \sum_{j=0}^{i} (i+1-j)\ell_j$$

Exercise 10.39. *Prove Theorem 10.38.*

Note that this implies the upper bounds we proved for the h_i's in Theorem 10.18, since $\sum_{j=0}^{i}(i+1-j)\ell_j$ is always nonnegative. Moreover, a combinatorial interpretation of the slack becomes evident.

Theorem 10.40.

$$e_k = \sum_{i=0}^{n} \binom{i}{k} (\ell_i - \ell_{i-1}) \quad \textit{with } \ell_{-1} = \ell_n = 1$$

Exercise 10.41. *Prove Theorem 10.40.*

Let us point out other counting problems which can be solved efficiently with the insights developed.

Exercise 10.42. *Given a ray* r *(emanating from point* q*) and* n *points* P *in the plane, design an efficient algorithm that counts the number of points connecting segments intersecting* r*. You may assume that* $P \cup \{q\}$ *is in general position and that* r *is disjoint from* P.

Exercise 10.43. *Let* w *be a line minus an interval on it (an infinite wall with a window). Given* n *points* P *in the plane, design an efficient algorithm that counts the number of point connecting segments disjoint from* w *(i.e. the number of pairs of points that see each other, either because they are both on the same side of* w *or because they see each other through the window in* w*. You may assume general position.*

Exercise 10.44. *Recall that a point* c *is a centerpoint of* P *if every halfplane containing* c *contains at least* $|P|/3$ *points in* P.

Identify the properties of e*,* h *and* l *that show that* 0 *is a centerpoint of* P.

Exercise 10.45. *Show that* $y_i = -y_{n-2-i}$ *and* $y_i \leqslant 0$ *for all* $0 \leqslant i \leqslant \lfloor \frac{n-2}{2} \rfloor$*. (We refer here to the* y_i*'s as defined by relations (10.3) at the beginning of this section. Hint: You may wish to recall Homework 10.35 and Exercise 10.33.)*

Exercise 10.46. *Show that* $h_i \geqslant h_{i-1}$ *for all* $0 \leqslant \lfloor \frac{n-3}{2} \rfloor$.

Questions

51. Explain how the h-vector of a planar point set is defined via a lifting. Give the relation between the e-vector (number of embracing k-sets) and the h-vector.

All of the following three questions also include Question 51.

52. Argue, why the h-vector is independent of the lifting.

53. Show how the ℓ-vector can be computed in $O(n \log n)$ time.

54. Argue why the ℓ-vector is symmetric ($\ell_i = \ell_{n-1-i}$ for all i, $0 \leqslant i \leqslant n-1$).